Examples of Constant Terms

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We attempt to put down some computations of constant terms of Eisenstein series. The key is the dark art of unfolding.

1 [Bum97] First Example

Consider the Eisenstein series for $\mathrm{Sl}_2(\mathbb{Z})$ with $z = x + iy \in \mathbb{C}$

$$E(z, s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{m,n\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{y^s}{|mz+n|^{2s}}$$

This satisfies the automorphie condition $\gamma \in \mathrm{Sl}_2(\mathbb{Z}), E(\gamma.z, s) = E(z, s)$.

Consider the Fourier expansion of E given by

$$E(z, s) = \sum_{r\in\mathbb{Z}} a_r(y, s)e^{2\pi irx}$$

with Fourier coefficients

$$a_r(y, s) = \int_0^1 E(x + iy, s)e^{-2\pi irx}dx$$

the constant term is when $r = 0$ i.e.

$$
\begin{aligned}
a_0(y,s) &= \int_0^1 E(x+iy,s)dx \\
&= \int_0^1 \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{m,n\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)\frac{1}{2}\int_0^1 \sum_{m,n\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)\frac{1}{2}\int_0^1 \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{y^s}{|n|^{2s}} + \sum_{m,n\in\mathbb{Z}^2\ m\neq 0} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)\frac{1}{2}\int_0^1 2y^s \sum_{n\in\mathbb{N}\setminus\{0\}} n^{-2s} + \sum_{m,n\in\mathbb{Z}^2\ m\neq 0} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)\frac{1}{2}\int_0^1 2y^s\zeta(2s) + \sum_{m,n\in\mathbb{Z}^2\ m\neq 0} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)\frac{1}{2}\int_0^1 2y^s\zeta(2s) + 2\sum_{m\in\mathbb{N}_{>0},n\in\mathbb{Z}} \frac{y^s}{|mx+miy+n|^{2s}}dx \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s)\int_0^1 dx + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0},n\in\mathbb{Z}} \int_0^1 |mx+miy+n|^{-2s}dx \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0},n\in\mathbb{Z}} \int_0^1 ((mx+n)^2+(my)^2)^{-s}dx \\
(\star)\quad &= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0}} \sum_{n\ mod\ m} \int_{\mathbb{R}} ((mx+n)^2+(my)^2)^{-s}dx \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0}} \sum_{n\ mod\ m} \int_{\mathbb{R}} ((m(\alpha-\frac{n}{m})+n)^2+m^2y^2)^{-s}d(\alpha-\frac{n}{m}) \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0}} \sum_{n\ mod\ m} \int_{\mathbb{R}} ((m\alpha-n+n)^2+m^2y^2)^{-s}d\alpha \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0}} \sum_{n\ mod\ m} \int_{\mathbb{R}} (m^2\alpha^2+m^2y^2)^{-s}d\alpha \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s}\Gamma(s)y^s \sum_{m\in\mathbb{N}_{>0}} m^{-2s} \sum_{n\ mod\ m} \int_{\mathbb{R}} (\alpha^2+y^2)^{-s}d\alpha \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \sum_{m\in\mathbb{N}_{>0}} m^{1-2s}\pi^{-s}\Gamma(s)y^s \int_{\mathbb{R}} (\alpha^2+y^2)^{-s}d\alpha
\end{aligned}
$$

We use that the absolute value in the sum means that there is a symmetry between n and $-n$, as well as pairs $(-m,-n)$ and (m,n). The identity in $(\star)$ is somewhat tricky although one can see it geometrically by drawing the pictures of $mx+n$ for different m,n pairs. Effectively the lines across all of $\mathbb{R}$ are cut up into length one segments and put in the interval $[0,1]$ and this gives the transition from the second to the first sum. Now we restrict to $re(s) > 1/2$ which allows us to equate the right hand side [Bum97][pg. 67 (6.8)]:

$$
\begin{aligned}
a_0(y,s) &= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \sum_{m\in\mathbb{N}_{>0}} m^{1-2s}\pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s} \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s}\sum_{m\in\mathbb{N}_{>0}} m^{1-2s} \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s}\zeta(2s-1) \\
&= \pi^{-s}\Gamma(s)y^s\zeta(2s) + \pi^{s-1}\Gamma(1-s)y^{1-s}\zeta(2-2s)
\end{aligned}
$$

Using functional equations of the zeta function.

2 General Unfolding

Theorem ([Gar] 5.2, [Fol16] Thm 2.49)**.** *Let $H \leq G$ be a closed subgroup. If $H \setminus G$ has a right G invariant measure (iff their modular functions agree on H)then the integral is unique up to scalar, namely for a given Haar measures dh on H and dg on G there is a unique invariant measure dq on $H \setminus G$ such that for all $f \in C_c^0(G)$*

$$\int_{H\setminus G} \int_H f(hq) dh dq = \int_G f(g) dg$$

Note that this quotient may not be a group, becuase H is not required to be normal.

So in particular under some mild hypotheses, with $\Gamma_\infty \leq \Gamma \leq G$ subgroups we have that

$$\int_{\Gamma\setminus G} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f \circ \gamma = \int_{\Gamma_\infty \setminus G} f$$

This can be applied in the setting of the constant term of an Eisenstein series, defined on $P = MN$, along alternalte parabolic $P' = M'N'$

$$\int_{N'(F)\setminus N'(\mathbb{A})} \sum_{\gamma \in P(K)\setminus G(K)} \varphi(\gamma n g) dn$$

by looking at the action of $N'(K)$ on $X := P(K) \setminus G(K)$. We know that a set is the disjoint union of its orbits under such an action so

$$X \cong \coprod_{i \in I} N'(\mathbb{A}) x_i$$

for some set of representatives $x_i \in X$. From the proof of the orbit stabiliser theorem we also have that

$$N'(\mathbb{A}) x_i \cong Stab(x_i) \setminus N'(K)$$

So applying the theorem

$$\begin{aligned}
\int_{N'(F)\setminus N'(\mathbb{A})} \sum_{\gamma \in P(K)\setminus G(K)} \varphi(\gamma n g) dn &= \int_{N'(F)\setminus N'(\mathbb{A})} \sum_{\gamma \in \coprod_i Stab(x_i)\setminus N'(K)} \varphi(\gamma n g) dn \\
&= \int_{N'(F)\setminus N'(\mathbb{A})} \sum_i \sum_{\gamma \in Stab(x_i)\setminus N'(K)} \varphi(\gamma n g) dn \\
&= \sum_i \int_{N'(F)\setminus N'(\mathbb{A})} \sum_{\gamma \in Stab(x_i)\setminus N'(K)} \varphi(\gamma n g) dn \\
&= \sum_i \int_{Stab(x_i)\setminus N'(\mathbb{A})} \varphi(n g) dn
\end{aligned}$$

3 [Bum97] Second Example

Let $G = \mathrm{Gl}_2$ and B be the standard Borel of upper triangular matricies, both defined over a global field F. Then the Eisenstein series

$$E(g, f) = \sum_{\gamma \in B(F) \backslash G(F)} f(\gamma . g)$$

is a map $\pi(\chi_1, \chi_2) \to \mathcal{A}(G(F) \setminus G(\mathbb{A}), \chi_1\chi_2)$ in the f variable. What these spaces are is not important for the calculation of the constant term, as long as you beleive that one exists.

Bump shows using Fourier inversion formula that the fourier expansion of this function is

$$E(g, f) = \sum_{\alpha \in F} c_\alpha(g, f)$$

where

$$c_\alpha(g, f) \ := \ \int_{A/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx$$

where ψ is a(n arbitrary non-trivial) character of $\mathbb{A}/F$. Thus the constant term is when $\alpha = 0$, this can readily be seen to be the definition of the Adelic constant term too, evaluated on the Levi :

$$\begin{aligned} c_0(g, f) &= \int_{A/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right) \psi(0) dx \\ &= \int_{A/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right) dx \\ &= \int_{A/F} \sum_{\gamma \in B(F) \backslash G(F)} f\left(\gamma . \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx \end{aligned}$$

Lemma. *The set $B(F) \setminus G(F)$ has a complete set of representatives given by*

$$\begin{pmatrix} & -1 \\ 1 & x \end{pmatrix}, \quad x \in F$$

and the identity.

Using this we will break up the sum as before.

$$\begin{aligned} c_0(g, f) &= \int_{A/F} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) + \sum_{\lambda \in F - \{0\}} f\left(\begin{pmatrix} & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx \\ &= \int_{A/F} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) + \sum_{\lambda \in F - \{0\}} f\left(\begin{pmatrix} & -1 \\ 1 & x + \lambda \end{pmatrix} g\right) dx \\ &= \int_{A/F} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx + \sum_{\lambda \in F - \{0\}} \int_{A/F} f\left(\begin{pmatrix} & -1 \\ 1 & x + \lambda \end{pmatrix} g\right) dx \\ &= f(g) + \int_{A} f\left(\begin{pmatrix} & -1 \\ 1 & x + \lambda \end{pmatrix} g\right) dx \end{aligned}$$

Ok I guess the idea is that this is easier to work with. The main reason being that we have removed the infinite sum of the Eisenstein series (even though there is a bigger infinite sum in the integral).

4 A General Computation of the Constant Term

This is a well known theorem, the explainations given by the references vary from very helpful ([GH24][Prop 10.4.2]) to none at all ([MW95][II.1.7]). They are still insufficient for someone like me to not spend a month confused about them.

We will use the following Lemmas to give a simplified expression of the constant term of an Eisenstein series. First fix $P = MN$ and $P' = M'N'$ two standard parabolics of suitable group G over F, with $E(x, \varphi, \lambda)$ defined via parabolic induction from P.

Lemma.
$$P(F) \setminus G(F) \cong \coprod_{w \in W_{M'} \setminus W_G / W_M} P'(F) \cap wP(F)w^{-1} \setminus P'(F)$$

Proof. Consider the Bruhat decomposition:

$$G(F) = \coprod_{w \in W_{M'} \setminus W_G / W_M} P(F)w^{-1}P'(F)$$

then
$$P(F) \setminus G(F) = \coprod_{w} P(F) \setminus P(F)w^{-1}P'(F)$$

so we analyse the summands, by the second isomorphism theorem we have a bijection

$$P(F) \setminus P(F)w^{-1}P'(F) \cong P(F) \cap P'(F) \setminus w^{-1}P'(F)$$

now if $[w^{-1}p] \in P(F) \cap P'(F) \setminus w^{-1}P'(F)$ then its $pw^{-1}p'$ for some $p \in P(F) \cap P'(F)$ and hence multiplying by w, in particular an isomorphism, gives $wpw^{-1}p' \in wP(F)w^{-1} \times P'(F)$ and so

$$w(P(F) \cap P'(F) \setminus w^{-1}P'(F)) \cong wP(F)w^{-1} \cap P'(F) \setminus P'(F)$$

Lemma. *Let $m', n' \in M'(F) \times N'(F)$ then*

$$m'n' \in wP(F)w^{-1} \iff m' \in wP(F)w^{-1}, \quad n' \in (m')^{-1}wP(F)w^{-1}m'$$

Proof. The forward implication is stated in [GH24], the converse follows from some algebra: First let $m' = wp_1w^{-1}$ and $n' = (m')^{-1}wp_2w^{-1}m'$ then

$$\begin{aligned} m'n' &= (wp_1w^{-1})^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}w^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}p_2p_1w^{-1} \in wP(F)w^{-1} \end{aligned}$$

Taking the contrapositive of this lemma will be used below. This is becuase our sums will be over quotients like $A \setminus B$ and therefore summing over the "elements" in B that are not in A; by our lemma would be the same as summing over two different such quotients. Now consider the computation:

$$
\begin{aligned}
E_{P'}(x,\varphi,\lambda) &= \int_{N'(F)\backslash N'(\mathbb{A})} E(nx,\varphi,\lambda)dn \\
([N'] \ := \ N'(F)\setminus N'(\mathbb{A})) &= \int_{[N']} \sum_{\delta\in P(F)\backslash G(F)} \varphi(\delta nx)dn \\
(\text{Lemma } 1) &= \int_{[N']} \sum_{\delta\in\coprod_{w\in W_{M'}\backslash W_G/W_M}} \sum_{P'(F)\cap wP(F)w^{-1}\backslash P'(F)} \varphi(\delta nx)dn \\
&= \sum_{w\in W_{M'}\backslash W_G/W_M} \int_{[N']} \sum_{p'\in P'(F)\cap wP(F)w^{-1}\backslash P'(F)} \varphi(w^{-1}p'nx)dn \\
(\text{Lemma } 2) &= \sum_{w} \sum_{m'\in M'(F)\cap wP(F)w^{-1}\backslash M'(F)} \int_{[N']} \sum_{n'\in N'(F)\cap(m')^{-1}wP(F)w^{-1}m'\backslash N'(F)} \varphi(w^{-1}m'n'nx)dn \\
(\text{Change Var}) &= \sum_{w}\sum_{m'} \int_{[N']} \sum_{n'\in N'(F)\cap wP(F)w^{-1}\backslash N'(F)} \varphi(w^{-1}n'nm'x)dn \\
(\text{Unfold}) &= \sum_{w}\sum_{m'} \int_{N'(F)\cap wP(F)w^{-1}\backslash N'(\mathbb{A})} \varphi(w^{-1}nm'x)dn
\end{aligned}
$$

The change of variables is $(m',n')\mapsto((m')^{-1}n'm',(m')^{-1}n'm')$. Again we assume that our x is sufficiently large so all the integrals converge.

5 Many Notions of Constant Term

For locally compact abelian groups we have the classical theory of Fourier analysis almost one to one. For compact non-abelian groups we get the Peter-Weyl formula, which is a straightforward analogue of the classical case. For reductive groups over an algebraic number field we have the notion of a constant term as defined in [MW95] for adelic automorphic forms. The goal here is to relate these things. My first idea is that the constant term comes from the decomposition of L2 in terms of sums of psuedo eisenstein series. So this should then be the coefficient of the "trivial" Eisenstein series. Aesthetically they are obviously all very similar.

References

[Bum97] Daniel Bump. *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.

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[MW95] C. Moeglin and J. L. Waldspurger. *Spectral Decomposition and Eisenstein Series: A Paraphrase of the Scriptures.* Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1995.